arXiv:physics/0702088v1 [physics.class-ph] 11 Feb 2007

Mechanics of Solids
Vol. 40, No. 6

ELASTOPLASTIC EQUATIONS WITH EXACT CONSISTENCY OF STRAINS FOR THE ELASTIC STRAIN FRAME AND THE VELOCITIES OF POINTS

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Abstract

We obtain an exact strain consistency equation for full, elastic and plastic strain characteristics that have a clear physical meaning and are naturally related to stresses. The dynamic equations are represented in a form that does not use the objective stress rate. This obviates a number of essential difficulties in the theory of finite elastoplastic strains.

1 Introduction

The present paper gives a detailed justification of the results obtained in [1]. To characterize them, let us very roughly outline the main difficulties encountered in mathematical elastoplasticity when passing from infinitesimal to finite strains and the methods used so far to overcome these difficulties. A detailed analysis can be found in the critical survey [2] of the state of the art in the theory by 1990 and in the monograph [3], which summarizes what has been done in the last fifty years.

Since the above-mentioned difficulties already arise when dealing with homogeneous isotropic elastic–perfectly plastic materials, we momentarily restrict ourselves to such materials.

In the Eulerian variables, elastoplastic deformation processes are described in terms of mass point velocities, material density, selected measures (characteristics) of full, elastic, and plastic strains, and stresses. With the advent in 1930 of the Prandtl–Reuß equation (see [4]), the closed system of elastoplasticity equations for infinitesimal strains was obtained. We single out the following three of these equations (the remaining three equations remain unchanged under the transition to finite strains):

A. The strain consistency equation (a tensor equation) relates full, elastic, and plastic strain characteristics.

B. The stress response function (a tensor equation) expresses the stress via an elastic strain characteristic.

C. The flow law (a tensor equation) relates the plastic strain to the stresses.

Under the transition to finite strains, the strain consistency equation (A) is not true in general ([3], 7.2) and the system is no longer closed. Moreover, not only measures but also the very notions of elastic and plastic strain have no precise physical meaning, and there is controversy (as yet unsettled) as to how exactly they should be introduced ([2], 4A).

In this connection, there has been quite a few general theories (some of which are presented in [5–11]), a fact that alone shows how unsatisfactory the situation is. All these theories can be divided into four groups.

1. Elastic strain is not considered at all. The plastic strain is introduced by a constitutive equation of type (C) via a measure that does not have a precise physical meaning. Two variables (plastic and full strain) are introduced in the constitutive equation (B) instead of elastic strain, and the system again becomes closed (e.g., see [5]).

D. The fact that the strain measure is artificial casts doubt on the existence (and all the more, the simplicity) of the constitutive equations (B) and/or (C).

2. The plastic strain is not considered, and so equation (C) disappears. Instead of (A), using various analogies, one introduces a constitutive differential equation for the elastic strain, which closes the system. It is this new equation that is doubtful (e.g., see [6, 7]).

3. Both elastic and plastic strains are introduced. One of them is determined via the other (using the full strain) but has no precise physical meaning.

In this case, the very definition already gives the strain consistency equation instead of (A). The system remains closed but has the drawback described above in (D) (e.g., see [8, 9]).

4. For the strain consistency equation (A), one uses the multiplicative decomposition of the full strain gradient in terms of the elastic and plastic strain gradients or the related additive decomposition of the full strain rate in the elastic and plastic strain rates [10].

Serious objections to the first relation can be found in [2], 4A. The second relation is not true in general and involves the objective stress rate, resulting in heavy inconsistencies ([2], 4F; [3], 7.2, 7.3).

In the present paper, elastic strain is characterized by the elastic strain frame $\breve{a}$, which is more informative than the Cauchy–Lagrange tensor. In the deformed medium, the elastic strain frame field generates the Riemannian metric of the "natural, unstressed state." Under sufficiently wide assumptions, this metric permits one to judge about the natural measure of deformed material objects (i.e., the measure that they would have after unloading), without resorting to actual unloading.

We introduce the plastic strain tensor, which has the same physical meaning in terms of the natural measure as does the elastic strain tensor for elastic media in terms of the geometric measure.

We obtain the strain consistency equation in frame form, providing an exact relationship between the full, elastic, and plastic strain characteristics.

We introduce the relative stress tensor (with respect to the Riemannian metric generated by an arbitrary nondegenerate frame field), whose restrictions are the natural stress tensor and the conventional Cauchy stress tensor and Piola–Kirchhoff second stress tensor.

For media homogeneous and isotropic in the natural state, we suggest a plastic constitutive law specified by a scalar function p of the principal stresses and possibly by parameters determined by the strain history. The presence or absence of yield surfaces is determined by the properties of p.

Under the assumptions accepted in the paper, we obtain a system of quasilinear first-order partial differential equations in Eulerian variables for $\boldsymbol{v}$ and $\breve{a}$, describing the elastoplastic deformation process in the entire space–time domain occupied by the medium. All equations are given in the form solved for the total time derivative, which makes the system suitable for numerical solution.[1]

2 Some notation and conventions

We assume that the real space in which the continuous medium moves is the three-dimensional Euclidean affine space. We denote it by E and the associated vector space by $\boldsymbol{E}$. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ be the scalar product and the corresponding norm in $\boldsymbol{E}$.

A frame consisting of vectors $\boldsymbol{a}_i \in \boldsymbol{E}$, $i = 1, 2, 3$, will be denoted by $\breve{a}$ and written as a column matrix, so that $\breve{a} := (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)^T$, where T stands for transposition. (For convenience, the entries of a row matrix are separated by commas.) In the presence of an ambiguity, we always take a nondegenerate right frame.The symbols := and =: stand for " is by definition equal to", where the object to be defined is written next to the colon.

Let us define the norm of a frame in E by the formula $\|\breve{a}\| := (|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2 + |\boldsymbol{a}_3|^2)^{1/2}$ and the difference of frames by the formula $\breve{a} - \breve{b} := (\boldsymbol{a}_1 - \boldsymbol{b}_1, \boldsymbol{a}_2 - \boldsymbol{b}_2, \boldsymbol{a}_3 - \boldsymbol{b}_3)^T$. The coefficients (contravariant coordinates) in the expansion of a vector $\boldsymbol{x} \in \boldsymbol{E}$ in the vectors of a frame $\breve{a}$ will be denoted by $\boldsymbol{x}_i^{\breve{a}}$, or, more briefly, by $x_i^{\breve{a}}$; $\boldsymbol{x}^{\breve{a}} := x^{\breve{a}} := (x_1^{\breve{a}}, x_2^{\breve{a}}, x_3^{\breve{a}})^T$. Thus $\boldsymbol{x} = x_i^{\breve{a}} \boldsymbol{a}_i$. Here and in what follows, summation from 1 to 3 is assumed over a repeated nonunderlined index in a monomial.

The scalar product defined on $\boldsymbol{E}$ can be represented in coordinate form in

[1] Note that the elastic strain frame was introduced for elastic media in I. A. Solomeshch and V. Sh. Khalilov, Nonlinear Elasticity Equations in Strain Frame Components [in Russian], No. 5089-B90, VINITI, Moscow, 1990 and for elastoplastic media in I. A. Solomeshch and M. A. Solomeshch, Dynamic Elastoplasticity Equations for the Elastic Strain Frame and the Velocities of Points [in Russian], No. 214-B95, VINITI, Moscow, 1995. The latter paper indicates the possible connection of the elastic strain frame in polycrystal materials with the strain of the crystal lattice. In the earlier-mentioned papers [6, 7, 9], frames actually coinciding with the elastic strain frame are introduced under various names (on the average, the trivector of microstructure variables) to characterize the elastic strain in such materials.

any frame $\check{e}$ orthonormal with respect to this product:

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \alpha_i^{\check{e}} \beta_i^{\check{e}} \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{E}. \tag{2.1}$$

Conversely, for an arbitrary nondegenerate frame $\check{a}$ one can construct a scalar product (generally different from $\langle \cdot, \cdot \rangle$) in $\boldsymbol{E}$ by setting

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\check{a}} := \alpha_i^{\check{a}} \beta_i^{\check{a}}; \tag{2.2}$$

the frame $\check{a}$ is orthonormal with respect to this scalar product. The Euclidean affine space obtained by the replacement of the scalar product $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle_{\check{a}}$ will be denoted by $E_{\check{a}}$. Obviously, $E_{\check{e}} = E$ if the frame $\check{e}$ is orthonormal in E.

The modulus of a vector $\boldsymbol{\alpha} \in \boldsymbol{E}$ in $E_{\check{a}}$ is equal to

$$|\boldsymbol{\alpha}|_{\check{a}} := \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle_{\check{a}}^{1/2}. \tag{2.3}$$

The volume of the parallelepiped spanned by a frame $\check{b}$ in the space E will be denoted by $|\check{b}|$, and the volume of the same parallelepiped in $E_{\check{a}}$ will be denoted by $|\check{b}|_{\check{a}}$.

Nondegenerate frames $\check{a}$ and $\check{a}'$ generating the same scalar product, i.e., such that $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\check{a}} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\check{a}'}$ for any $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \boldsymbol{E}$, will be called *equivalent*.

For frames $\check{a}$ and $\check{b}$, by $\check{a}^{\check{b}}$ we denote the matrix whose jth column is $a_j^{\check{b}}$. Matrices are sometimes indicated by the symbol $\hat{}$. For column matrices $\hat{c} := (c_1, c_2, c_3)^T$ and $\hat{d} := (d_1, d_2, d_3)^T$ with numerical entries, we write

$$(\hat{c}, \hat{d}) := c_i d_i, \quad |\hat{c}| := (\hat{c}, \hat{c})^{1/2} \tag{2.4}$$

Let a continuous medium occupying an open spatial domain $\mathrm{V} \subset E$ at the initial time t_0 fill a domain V_t at the final time t. The spatial position of a particle M at time t_0 will be denoted by x (the Lagrangian variable), and the position of the same particle at time t will be denoted by x (the Eulerian variable).

The function $x = x(\mathrm{x})$ specifying the correspondence between the initial and final positions of a mass point, as well as the laws of motion considered in what follows, is assumed to be one-to-one and have an invertible derivative x'_{x}.

In general, we adhere to the following principle whenever possible: objects characterizing the medium at the terminal time are denoted by italic Latin letters, and the corresponding objects at the initial time are denoted by the same letters typeset in Roman; any function of a particle expressed via x or x is denoted by same letter with the argument indicated where necessary; notation introduced for some object is later used for other objects of the same nature. Unless explicitly specified otherwise, we assume that all functions occurring in the text (including the derivatives) are continuous, all curves and surfaces are smooth, and all sets over which integration is carried out are measurable and closed. Whenever the metric is not indicated explicitly, the metric of the main space E is meant.

In the following, to reveal elastic and plastic strains of the medium, we need to compare the measure of material objects in Riemannian spaces that are different in general. The corresponding mathematical technique is given in the following section.

3 The change in the measure of geometric objects under a mapping of Riemannian affine spaces

3.1. The Riemannian metric generated by a frame field. Let V be an open domain in an affine space E. (Momentarily, it is not important whether E is Euclidean.) A Riemannian metric on V is usually defined with the help of a metric form

$$g_{ij}(x(\xi))d\xi_i d\xi_j \tag{3.1}$$

in some curvilinear coordinate system $x = x(\xi)$. Essentially, formula (3.1) defines a scalar product at any point $x \in V$ (more precisely, on the tangent space at x, i.e., in our case, on $\boldsymbol{E}$). Therefore, one can define a Riemannian metric by directly specifying a scalar product at the points of V.

For our purposes, it is convenient to specify each of these products by specifying a frame orthonormal with respect to this product.

Thus we adopt the following scheme. Let a nondegenerate frame field $\breve{\alpha}(x)$ be given on V. For the scalar product at a point x we take the product $\langle\cdot,\cdot\rangle_{\breve{\alpha}(x)}$, with respect to which $\breve{\alpha}(x)$ is orthonormal (see (2.2)). The Riemannian space thus defined is said to be generated by the frame field $\breve{\alpha}(x)$ and will be denoted by $V_{\breve{\alpha}}$. The metric in it will be called the $\breve{\alpha}$-metric.

The relationship between the coefficients of the metric form and $\breve{\alpha}(x)$ is as follows:

$$g_{ij}(x(\xi)) = \langle \boldsymbol{k}_i(\xi), \boldsymbol{k}_j(\xi)\rangle_{\breve{\alpha}(x(\xi))}$$

where

$$\breve{k}(\xi) := \frac{\partial x}{\partial \xi} := \left(\frac{\partial x}{\partial \xi_1}, \frac{\partial x}{\partial \xi_2}, \frac{\partial x}{\partial \xi_3}\right)^T. \tag{3.2}$$

Note that $\breve{k}(\xi)$ is the local frame of the curvilinear coordinate system at the point $x(\xi)$.

The conventional formulas of Riemannian geometry in terms of the frame $\breve{\alpha}(x)$ acquire the following form. The cosine of the angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ at a point x is

$$\cos(\widehat{\boldsymbol{x},\boldsymbol{y}})_{\breve{\alpha}(x)} := \frac{\langle \boldsymbol{x},\boldsymbol{y}\rangle_{\breve{\alpha}(x)}}{|\boldsymbol{x}|_{\breve{\alpha}(x)} \cdot |\boldsymbol{y}|_{\breve{\alpha}(x)}}; \tag{3.3}$$

The length of a curve l: $x = x(\tau)$, $\tau \in [\tau_0, \tau_1]$, is

$$|l|_{\breve{\alpha}} := \int_{\tau_0}^{\tau_1} |x'(\tau)|_{\breve{\alpha}(x(\tau))}\, d\tau; \tag{3.4}$$

The area of a surface s: $x = x(u)$, $u = (u_1, u_2)^T \in \sigma \subset \boldsymbol{R}^2$, is

$$|s|_{\breve{\alpha}} = \int_\sigma \sqrt{|x'_{u_1}|^2_{\breve{\alpha}(x)} \cdot |x'_{u_2}|^2_{\breve{\alpha}(x)} - \langle x'_{u_1}, x'_{u_2}\rangle^2_{\breve{\alpha}(x)}}\, du; \tag{3.5}$$

The volume of a domain v: $x = x(\xi)$, $\xi = (\xi_1, \xi_2, \xi_3)^T \in \omega \subset \boldsymbol{R}^3$, is

$$|v|_{\breve{\alpha}} = \int_{\omega} |\breve{k}(\xi)|_{\breve{\alpha}(x(\xi))}\, d\xi. \tag{3.6}$$

If $\breve{\alpha}(x)$ is a constant $\breve{\alpha}_0$, then $V_{\breve{\alpha}_0}$ is the Euclidean space with scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\breve{\alpha}_0}$.

3.2. The dilatation coefficients under a change of the Riemannian metric. Let $\breve{\alpha}(x)$ and $\breve{\beta}(x)$ be nondegenerate frame fields defined on V, and let $V_{\breve{\alpha}}$ and $V_{\breve{\beta}}$ be the Riemannian spaces generated by these frame fields. Let us find the dilatation coefficients for the measures of one- to three-dimensional objects under the transition from their measurement in the $\breve{\alpha}$-metric to the measurement in the $\breve{\beta}$-metric.

The length dilatation coefficient. Take an arbitrary vector $\boldsymbol{m} \neq 0$. Let l: $x = x(\tau)$, $\tau \in [\tau_0, \tau_1]$ be an arbitrary curve issuing from the point x_0, let $x(\tau_0) = x_0$; $\boldsymbol{m}$ be the tangent vector to l at x_0, and let $l_{\tau^*} := x([\tau_0, \tau^*])$ for each $\tau^* \in (\tau_0, \tau_1]$.

The length dilatation coefficient at the point x_0 in the direction $\boldsymbol{m}$ is computed with the use of (3.4):

$$K_1(x_0, \boldsymbol{m}, \breve{\alpha}, \breve{\beta}) := \lim_{\tau^* \to \tau_0} \frac{|l_{\tau^*}|_{\breve{\beta}}}{|l_{\tau^*}|_{\breve{\alpha}}} = \lim_{\tau^* \to \tau_0} \frac{\int_{\tau_0}^{\tau^*} |x'_\tau|_{\breve{\beta}} d\tau}{\int_{\tau_0}^{\tau^*} |x'_\tau|_{\breve{\alpha}} d\tau} = \frac{|x'_\tau(\tau_0)|_{\breve{\beta}(x_0)}}{|x'_\tau(\tau_0)|_{\breve{\alpha}(x_0)}}\,.$$

The vectors $x'(\tau_0)$ and $\boldsymbol{m}$ are collinear, and hence $x'_\tau(\tau_0) = c\boldsymbol{m}$ for some c. Eventually, we obtain

$$K_1(x_0, \boldsymbol{m}, \breve{\alpha}, \breve{\beta}) = \frac{|\boldsymbol{m}|_{\breve{\beta}(x_0)}}{|\boldsymbol{m}|_{\breve{\alpha}(x_0)}}. \tag{3.7}$$

The volume dilatation coefficient. Take an arbitrary $\forall x_0 \in V$. Let v: $x = x(\xi)$, $\xi \in \omega \subset \mathbf{R^3}$, be an arbitrary three-dimensional subdomain of V containing x_0, $x_0 = x(\xi_0)$; suppose that $\omega^* \subset \omega$ contains ξ_0 and $v_{\omega^*} := x(\omega^*)$.

The volume dilatation coefficient at the point x_0 is computed with the use of (3.6):

$$K_3(x_0, \breve{\alpha}, \breve{\beta}) := \lim_{\omega^* \to \xi_0} \frac{|v_{\omega^*}|_{\breve{\beta}}}{|v_{\omega^*}|_{\breve{\alpha}}} = \lim_{\omega^* \to \xi_0} \frac{\int_{\omega^*} |\breve{k}(\xi)|_{\breve{\beta}(x(\xi))}\, d\xi}{\int_{\omega^*} |\breve{k}(\xi)|_{\breve{\alpha}(x(\xi))}\, d\xi} = \frac{|\breve{k}(x_0)|_{\breve{\beta}(x_0)}}{|\breve{k}(x_0)|_{\breve{\alpha}(x_0)}}. \tag{3.8}$$

The volume in $V_{\breve{\beta}(x_0)}$ of the parallelepiped spanned by $\breve{k}(x_0)$ is equal to $|\breve{k}(x_0)|_{\breve{\beta}(x_0)} = |\breve{k}^{\breve{\beta}}|$, and similarly, $|\breve{k}(x_0)|_{\breve{\alpha}(x_0)} = |\breve{k}^{\breve{\alpha}}|$.

We denote $\breve{\alpha}^{\breve{\beta}} =: A$; then $\breve{\alpha} = A^T \breve{\beta}$, and hence the $\breve{\alpha}$- and $\breve{\beta}$-coordinates of an arbitrary vector $\boldsymbol{y}$ are related by $y^{\breve{\alpha}} = A^{-1} y^{\breve{\beta}}$. Therefore, $|\breve{k}|_{\breve{\alpha}} = |\breve{k}^{\breve{\alpha}}| = |A^{-1}\breve{k}^{\breve{\beta}}| = |A|^{-1} \cdot |\breve{k}|_{\breve{\beta}}$. Substituting this into (3.8), we obtain

$$K_3(x_0, \breve{\alpha}, \breve{\beta}) = |A| = |\breve{\alpha}(x_0)|_{\breve{\beta}(x_0)}. \tag{3.9}$$

The area dilatation coefficient. Take an arbitrary $\forall x_0 \in V$. Let s: $x = x(u)$, $u = (u_1, u_2) \in \sigma \subset \mathbf{R}^2$, be an arbitrary oriented surface in V with unit positive $\breve{\beta}$-normal $\boldsymbol{n}$ at the point x_0, $x_0 = x(u_0)$; we assume that $\sigma^* \subset \sigma$ contains u_0 and $s_{\sigma^*} := x(\sigma^*)$. The area dilatation coefficient at the point x_0 is computed with the use of (3.5):

$$K_2(x_0, \boldsymbol{n}, \breve{\alpha}, \breve{\beta}) := \lim_{\sigma^* \to u_0}(|s_{\sigma^*}|_{\breve{\beta}}/|s_{\sigma^*}|_{\breve{\alpha}})=$$

$$= \lim_{\sigma^* \to u_0} \frac{\int\limits_{\sigma^*} \sqrt{|x'_{u_1}|^2_{\breve{\beta}(x)} \cdot |x'_{u_2}|^2_{\breve{\beta}(x)} - \langle x'_{u_1}, x'_{u_2}\rangle^2_{\breve{\beta}(x)}}du}{\int\limits_{\sigma^*} \sqrt{|x'_{u_1}|^2_{\breve{\alpha}(x)} \cdot |x'_{u_2}|^2_{\breve{\alpha}(x)} - \langle x'_{u_1}, x'_{u_2}\rangle^2_{\breve{\alpha}(x)}}du} =$$

$$= \left(\frac{\sqrt{|x'_{u_1}|^2_{\breve{\beta}(x)} \cdot |x'_{u_2}|^2_{\breve{\beta}(x)} - \langle x'_{u_1}, x'_{u_2}\rangle^2_{\breve{\beta}(x)}}}{\sqrt{|x'_{u_1}|^2_{\breve{\alpha}(x)} \cdot |x'_{u_2}|^2_{\breve{\alpha}(x)} - \langle x'_{u_1}, x'_{u_2}\rangle^2_{\breve{\alpha}(x)}}} \right)_{|x=x_0} \tag{3.10}$$

From this formula, we conclude that the dilatation coefficient is same as if
1. $\breve{\beta}(x) \equiv \breve{\beta}(x_0)$ and $\breve{\alpha}(x) \equiv \breve{\alpha}(x_0)$,, i.e., both metrics are Euclidean.
2. s lies in the plane Π with positive unit $\breve{\beta}$-normal $\boldsymbol{n}$ at the point x_0.

But in this case the integrands in (3.10) are constants and there is no need to pass to the limit; i.e.,

$$K_2(x_0, \boldsymbol{n}, \breve{\alpha}, \breve{\beta}) = \frac{|s|_{\breve{\beta}(x_0)}}{|s|_{\breve{\alpha}(x_0)}} \quad \forall s \subset \Pi. \tag{3.11}$$

Consider the parallelepiped v spanned by the vectors $\boldsymbol{m}$, $\boldsymbol{l}$ (parallel to Π), and $\boldsymbol{n}$ forming a nondegenerate right frame at x_0. We denote the parallelogram spanned by $\boldsymbol{m}$ and $\boldsymbol{l}$ by s and the positive unit $\breve{\alpha}$-normal to Π by $\mathbf{n}$,

$$|v|_{\breve{\beta}} = |s|_{\breve{\beta}} \cdot |\boldsymbol{n}|_{\breve{\beta}} = |s|_{\breve{\beta}}, \quad |v|_{\breve{\alpha}} = |s|_{\breve{\alpha}} \cdot \langle \boldsymbol{n}, \mathbf{n}\rangle_{\breve{\alpha}}.$$

Therefore, in view of (3.11) and (3.9),

$$K_2(x_0, \boldsymbol{n}, \breve{\alpha}, \breve{\beta}) = \frac{|v|_{\breve{\beta}}}{|v|_{\breve{\alpha}}} \cdot \langle \boldsymbol{n}, \mathbf{n}\rangle_{\breve{\alpha}} = |\breve{\alpha}|_{\breve{\beta}} \cdot \langle \boldsymbol{n}, \mathbf{n}\rangle_{\breve{\alpha}}. \tag{3.12}$$

Let us find $\mathbf{n}$. To this end, we transform the equation of the plane Π in the $\breve{\beta}$-metric to the equation in the $\breve{\alpha}$-metric.

The equation Π in the $\breve{\beta}$-metric has the form

$$\langle \boldsymbol{n}, \boldsymbol{x}\rangle_{\breve{\beta}} = 0, \tag{3.13}$$

where $x - x_0 =: \boldsymbol{x}$, $\langle \boldsymbol{n}, \boldsymbol{x}\rangle_{\breve{\beta}} = (n^{\breve{\beta}}, x^{\breve{\beta}})$. $x^{\breve{\alpha}} = A^{-1}x^{\breve{\beta}}$, where $A = \breve{\alpha}^{\breve{\beta}}$. Therefore, $\langle \boldsymbol{n}, \boldsymbol{x}\rangle_{\breve{\beta}} = (n^{\breve{\beta}}, Ax^{\breve{\alpha}}) = (A^T n^{\breve{\beta}}, x^{\breve{\alpha}})$. If we now introduce the vector $\boldsymbol{N}$ with $\breve{\alpha}$-coordinates

$$N^{\breve{\alpha}} := A^T n^{\breve{\beta}}, \tag{3.14}$$

then $\langle \boldsymbol{n}, \boldsymbol{x}\rangle_{\breve{\beta}} = (N^{\breve{\alpha}}, x^{\breve{\alpha}}) = \langle \boldsymbol{N}, \boldsymbol{x}\rangle_{\breve{\alpha}}$.

By substituting this into (3.13) and by normalizing $\boldsymbol{N}$ in the $\check{\alpha}$-norm, we obtain the equation of Π in the $\check{\alpha}$-metric:

$$\langle \boldsymbol{N}/|N^{\check{\alpha}}|, \boldsymbol{x}\rangle_{\check{\alpha}} = 0.$$

Hence $\mathbf{n} = \boldsymbol{N}/|N^{\check{\alpha}}|$, and using (3.14), we obtain

$$\mathrm{n}^{\check{\alpha}} = \frac{A^T n^{\check{\beta}}}{|A^T n^{\check{\beta}}|}. \tag{3.15}$$

Now, using the expression of the $\check{\alpha}$-coordinates via the $\check{\beta}$-coordinates and (3.15), we have

$$\begin{aligned}\langle \boldsymbol{n}, \mathbf{n}\rangle_{\check{\alpha}} = (n^{\check{\alpha}}, \mathrm{n}^{\check{\alpha}}) &= (A^{-1} n^{\check{\beta}}, A^T n^{\check{\beta}})/|A^T n^{\check{\beta}}| \\ &= |\boldsymbol{n}|^2_{\check{\beta}}/|A^T n^{\check{\beta}}| = 1/|A^T n^{\check{\beta}}|.\end{aligned}$$

By substituting this into (3.12), we obtain

$$K_2(x_0, \boldsymbol{n}, \check{\alpha}, \check{\beta}) = \frac{|\check{\alpha}(x_0)|_{\check{\beta}(x_0)}}{|A^T n^{\check{\beta}}|}, \tag{3.16}$$

where $A = \check{\alpha}(x_0)^{\check{\beta}(x_0)}$.

4 The strain frame. The relative stresses

4.1. Material objects (curves, surfaces, three-dimensional regions) are deformed with respect to the initial state as a result of the displacement $x = x(\mathrm{x})$. One can characterize the resulting strain by comparing the measures (lengths, areas, volumes) of these objects, as well as the angles between material curves, at time t and the initial time t_0. Therefore, we characterize all above-mentioned objects in the strained state, i.e., at time t, both by their geometric measure (length, area, etc.) at time t and by their initial measure at time t_0.

Clearly, the specification of $x(\mathrm{x})$ or $x'(\mathrm{x})$ completely determines the terminal strain of the medium. In turn, the operator $x'(\mathrm{x}) =: \mathcal{A}(\mathrm{x})$ at each point is determined by the specification of an arbitrary nondegenerate frame $\check{\mathrm{a}}(\mathrm{x})$ and the frame

$$\check{a}(x) := (\mathcal{A}(\mathrm{x})\mathbf{a}_1(\mathrm{x}), \mathcal{A}(\mathrm{x})\mathbf{a}_2(\mathrm{x}), \mathcal{A}(\mathrm{x})\mathbf{a}_3(\mathrm{x}))^T =: \mathcal{A}(\mathrm{x})\check{\mathrm{a}}(\mathrm{x}); \tag{4.1}$$

one even does not need to require that the function $\check{\mathrm{a}}(\mathrm{x})$ is continuous.

However, if the frames $\check{\mathrm{a}}(\mathrm{x})$ are orthonormal, then, to specify the strain of the medium at time t, it suffices to define the frame $\check{a}(x)$ alone at each point $x \in V_t$. (Note that $\check{a}(x) = \check{a}(\mathrm{x})$ for $x = x(\mathrm{x})$; i.e., $\check{a}(x)$ and $\check{a}(\mathrm{x})$ specify the same function of a material point expressed via the Eulerian and Lagrangian coordinates, respectively.)

4.2. Let us prove this. The computation of the geometric measure of the deformed (i.e., given at time t) material objects does not encounter difficulties. Therefore, assuming that the frame field $\breve{a}(x)$ is given, we focus our attention on the computation of their initial measure.

Let l, l_i $(i = 1, 2)$, s, and v be oriented material curves, a surface, and a three-dimensional region given at the initial time in V; let $l := x(\mathrm{l})$, $l_i := x(\mathrm{l}_i)$, $s := x(\mathrm{s})$, and $v := x(\mathrm{v})$ be the same objects at the terminal time; we suppose that l_1 and l_2 meet at the point x_0; finally, $x_0 = x(\mathrm{x}_0)$; $\mathrm{x} = \mathrm{x}(\tau)$; $\mathrm{x} = \mathrm{x}_i(\tau)$, $\tau \in [\tau_1, \tau_2]$; $\mathrm{x} = \mathrm{x}(u)$, $u(u_1, u_2) \in \sigma \subset \mathbf{R}^2$, and $\mathrm{x} = \mathrm{x}(\xi)$, $\xi(\xi_1, \xi_2, \xi_3) \in \omega \subset \mathbf{R}^3$ are parametrizations of l, l_i, s, and v.

To denote the geometric measure of (length, area, or volume) of 1–3-dimensional objects, we shall use vertical bars. The initial measure of such objects given in the strained state will be denoted in the same way but with the subscript t. For example, $|\mathrm{l}|$ is the geometric length of l, $|v|$ is the geometric volume of v, and $|s|_t$ is the geometric area of s (i.e., $|\mathrm{s}|$). An geometric angle between vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ will be denoted by $(\widehat{\boldsymbol{n}, \boldsymbol{m}})$, and the initial angle between the same vectors issuing from a point x will be denoted by $(\widehat{\boldsymbol{n}, \boldsymbol{m}})_{x,t}$. The adjective "geometric" for a measure will usually be omitted.

First, let us compute the initial angle between the curves l_i at the point x_0. For the parametrization of l_i one can take $x = x(\mathrm{x}_i(\tau)) =: x_i(\tau)$. Let τ_0 be the parameter value corresponding to x_0 on the curves l_i; then it also corresponds to x_0 on l_i.

The derivatives $x_i'(\tau_0) =: \boldsymbol{m}_i$ are the positive tangent vectors to the curves l_i at the point x_0, and $\mathrm{x}_i'(\tau_0) =: \mathbf{m}_i$ is the tangent vector to l_i at the point x_0; here $\boldsymbol{m}_i$ and $\mathbf{m}_i$ are related by

$$\boldsymbol{m}_i = x_i'(\tau_0) = x_{\mathrm{x}}'(\mathrm{x}_0)\mathrm{x}_i'(\tau_0) = \mathcal{A}(\mathrm{x}_0)\mathbf{m}_i. \tag{4.2}$$

We need the following lemma.

Lemma. If $\mathcal{A}$ *is a linear invertible operator on* $\boldsymbol{E}$*,* $\check{\mathrm{a}}$ *is a nondegenerate frame, and* $\breve{a} = \mathcal{A}\check{\mathrm{a}}$*, then for any* $\boldsymbol{\alpha}$*,* $\boldsymbol{\beta} \in \boldsymbol{E}$ *and an arbitrary frame* $\check{b}$ *one has*

$$\alpha^{\check{\mathrm{a}}} = (\mathcal{A}\boldsymbol{\alpha})^{\breve{a}} \tag{4.3}$$

$$< \boldsymbol{\alpha}, \boldsymbol{\beta} >_{\check{\mathrm{a}}} = < \mathcal{A}\boldsymbol{\alpha}, \mathcal{A}\boldsymbol{\beta} >_{\breve{a}}, \quad |\boldsymbol{\alpha}|_{\check{\mathrm{a}}} = |\mathcal{A}\boldsymbol{\alpha}|_{\breve{a}} \tag{4.4}$$

$$|\check{b}|_{\check{\mathrm{a}}} = |\mathcal{A}\check{b}|_{\breve{a}} \tag{4.5}$$

Proof. We have $\boldsymbol{\alpha} = \alpha_i^{\check{\mathrm{a}}}\mathbf{a}_i$. Therefore, $\mathcal{A}\boldsymbol{\alpha} = \alpha_i^{\check{\mathrm{a}}}\mathcal{A}\mathbf{a}_i = \alpha_i^{\check{\mathrm{a}}}\boldsymbol{a}_i$; i.e., $(\mathcal{A}\boldsymbol{\alpha})^{\breve{a}} = \alpha^{\check{\mathrm{a}}}$. Assertions (4.4) readily follow from (4.3) by virtue of definitions (2.2) and (2.3). Further, using the Gram determinant, we have

$$|\check{b}|_{\check{\mathrm{a}}} = |\,\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle_{\check{\mathrm{a}}}|^{1/2} = |\,\langle \mathcal{A}\boldsymbol{b}_i, \mathcal{A}\boldsymbol{b}_j \rangle_{\breve{a}}|^{1/2} = |\mathcal{A}\check{b}|_{\breve{a}},$$

and the proof of the lemma is complete.

Using (2.1)–(2.3), we compute the cosine of the initial angle between the curves l_i at the point x_0:

$$\cos(\widehat{\boldsymbol{m}_1, \boldsymbol{m}_2})_{x_0,t} = \cos(\widehat{\mathbf{m}_1, \mathbf{m}_2}) = \frac{\langle \mathbf{m}_1, \mathbf{m}_2 \rangle}{|\mathbf{m}_1| \cdot |\mathbf{m}_2|} = \frac{\langle \mathbf{m}_1, \mathbf{m}_2 \rangle_{\check{\mathrm{a}}(\mathrm{x}_0)}}{|\mathbf{m}_1|_{\check{\mathrm{a}}(\mathrm{x}_0)} \cdot |\mathbf{m}_2|_{\check{\mathrm{a}}(\mathrm{x}_0)}}.$$

Thus, using (4.4) and (4.2), we obtain

$$\cos(\widehat{\boldsymbol{m}_1, \boldsymbol{m}_2})_{x_0,t} = \frac{\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle_{\check{a}(x_0)}}{|\boldsymbol{m}_1|_{\check{a}(x_0)} \cdot |\boldsymbol{m}_2|_{\check{a}(x_0)}}. \tag{4.6}$$

It is easy to express the initial length of the curve l via its equation

$$|l|_t = |\mathrm{l}| = \int_{\tau_1}^{\tau_2} |\mathrm{x}'(\tau)|\, d\tau = \int_{\tau_1}^{\tau_2} |\mathrm{x}'(\tau)|_{\check{\mathrm{a}}(\mathrm{x}(\tau))}\, d\tau.$$

Since $x'(\tau) = x'_{\mathrm{x}}(\mathrm{x}(\tau))\mathrm{x}'(\tau) = \mathcal{A}(x(\tau))\mathrm{x}'(\tau)$, we obtain, in view of (4.4),

$$|l|_t = \int_{\tau_1}^{\tau_2} |x'(\tau)|_{\check{a}(x(\tau))}\, d\tau. \tag{4.7}$$

In a similar way, we find the expressions for the initial area and volume:

$$|s|_t = |\mathrm{s}| = \int_\sigma \sqrt{\langle \mathrm{x}'_{u_1}, \mathrm{x}'_{u_1} \rangle \cdot \langle \mathrm{x}'_{u_2}, \mathrm{x}'_{u_2} \rangle - \langle \mathrm{x}'_{u_1}, \mathrm{x}'_{u_2} \rangle^2}\, du.$$

Let us replace the scalar product $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle_{\check{\mathrm{a}}(\mathrm{x})}$. (Recall that the frames $\check{\mathrm{a}}(\mathrm{x})$ are orthonormal.) Then we apply (4.4) and take into account the fact that $x'_{u_i} = x'_{\mathrm{x}}\, \mathrm{x}'_{u_i} = \mathcal{A}\mathrm{x}'_{u_i}$ to obtain

$$|s|_t = \int_\sigma \sqrt{\langle x'_{u_1}, x'_{u_1} \rangle_{\check{a}(x)} \cdot \langle x'_{u_2}, x'_{u_2} \rangle_{\check{a}(x)} - \langle x'_{u_1}, x'_{u_2} \rangle^2_{\check{a}(x)}}\, du. \tag{4.8}$$

The coordinate frame of the curvilinear coordinates $\mathrm{x} = \mathrm{x}(\xi)$ at the point $\check{\mathrm{k}}(\mathrm{x}) := \left(\frac{\partial \mathrm{x}}{\partial \xi_1}, \frac{\partial \mathrm{x}}{\partial \xi_2}, \frac{\partial \mathrm{x}}{\partial \xi_3}\right)^T =: \frac{\partial \mathrm{x}}{\partial \xi},$; therefore, $|v|_t = |\mathrm{v}| = \int_\omega |\check{\mathrm{k}}(\mathrm{x})|\, d\xi$. Since the frames $\check{\mathrm{a}}(\mathrm{x})$ are orthonormal, it follows that $E = E_{\check{\mathrm{a}}(\mathrm{x})}$, and, using (4.5), we obtain $|\check{\mathrm{k}}(\mathrm{x})| = |\check{\mathrm{k}}(\mathrm{x})|_{\check{\mathrm{a}}(\mathrm{x})} = |\mathcal{A}(\mathrm{x})\check{\mathrm{k}}(\mathrm{x})|_{\check{a}(x)}$. Since

$$\mathcal{A}(\mathrm{x})\check{\mathrm{k}}(\mathrm{x}) = x'_{\mathrm{x}}(\mathrm{x})\frac{\partial \mathrm{x}}{\partial \xi} = \left(x'_{\mathrm{x}}(\mathrm{x})\mathrm{x}'_{\xi_1}, x'_{\mathrm{x}}(\mathrm{x})\mathrm{x}'_{\xi_2}, x'_{\mathrm{x}}(\mathrm{x})\mathrm{x}'_{\xi_3}\right)^T =: \check{k}(x)$$

is a coordinate frame of the curvilinear coordinates $x = x(\xi)$ at the point x, we have

$$|v|_t = \int_\omega |\check{k}(x)|_{\check{a}(x)}\, d\xi. \tag{4.9}$$

Formulas (4.6)–(4.9) show that, given the frame field $\check{a}(x)$, one can compute the initial measure of 1–3-dimensional material objects and the initial angle between the material curves from the equations of these objects in the strained state (i.e., at time t). It follows that by specifying the field $\check{a}(x)$ one completely determines the strain of the medium at time t.

On the other hand, Eqs. (4.6)–(4.9) show (see Section 3.1) that the original metric for deformed material objects is the metric generated by the frame field $\check{a}(x)$, i.e., the metric obtained in the Riemannian affine space $V_{t,\check{a}}$ by the

replacement of the scalar product $\langle\cdot,\cdot\rangle$ common for all points in V_t by the field of scalar products $\langle\cdot,\cdot\rangle_{\breve{a}(x)}$, individual for each point $x\in V_t$.

4.3. We take a point $x\in V_t$ and a vector $\boldsymbol{m}\neq 0$. Let l be a material curve at time t for which $\boldsymbol{m}$ is the tangent vector at the point x. We consider the length dilatation coefficient for this curve at time t, equal to the ratio $|l|/|l|_t$ of its length at time t to its length in the initial state.

It turn out (see Section 3.2) that the limit of this coefficient as the curve shrinks along itself to the point x, is independent of the choice of l and depends only on x and $\boldsymbol{m}$. We refer to this limit as the length dilatation coefficient at the point x at time t (i.e., at the point (x,t)) in the direction $\boldsymbol{m}$ and denote it by $K_1(x,\boldsymbol{m})$.

The area dilatation coefficient $K_2(x,\boldsymbol{n})$ at a point (x,t) of a material surface with unit normal $\boldsymbol{n}$ at the point x at time t and the volume dilatation coefficient $K_3(x)$ at the point (x,t) are introduced in a similar way.

Since for each orthonormal frame $\breve{e}$ the scalar product $\langle\cdot,\cdot\rangle$ coincides with $\langle\cdot,\cdot\rangle_{\breve{e}}$, it follows that the geometric measure on V_t coincides with the measure on $V_{t,\breve{e}}$ for any orthonormal frame field $\breve{e}(x)$ on V_t. Furthermore, the initial metric coincides with the metric in $V_{t,\breve{a}}$. Therefore, K_i is the measure dilatation coefficient for the passage from the $\breve{a}$-metric to the $\breve{e}$-metric (i.e., the geometric metric) in V_t. Hence, according to formulas (3.7), (3.16), and (3.9) with $\breve{\alpha}=\breve{a}$ and $\breve{\beta}=\breve{e}$, we have

$$K_1(x,\boldsymbol{m})=\frac{|\boldsymbol{m}|}{|\boldsymbol{m}|_{\breve{a}}},\quad K_2(x,\boldsymbol{n})=\frac{|\breve{a}|}{|A^T n^{\breve{e}}|},\quad K_3(x)=|\breve{a}|,\tag{4.10}$$

where $A=\breve{a}^{\breve{e}}$.

Formula (4.6), together with (4.10), shows that $\breve{a}(x)$ completely determines the strain of the medium at the point (x,t).

Therefore, $\breve{a}(x)$ will be called the strain frame. Let us give a complete definition taking into account (4.1) and using derivatives with respect to a vector.

The strain frame of the medium at a point x at time t is defined as

$$\begin{aligned}&\breve{a}(x):=\frac{\partial x(\mathrm{x})}{\partial\breve{\mathrm{a}}}:=\left(\frac{\partial x(\mathrm{x})}{\partial\mathbf{a}_1},\,\frac{\partial x(\mathrm{x})}{\partial\mathbf{a}_2},\,\frac{\partial x(\mathrm{x})}{\partial\mathbf{a}_3}\right)^T=\left(x'(\mathrm{x})\mathbf{a}_1,\,x'(\mathrm{x})\mathbf{a}_2,\,x'(\mathrm{x})\mathbf{a}_3\right)^T\\&=x'(\mathrm{x})\breve{\mathrm{a}}\,,\end{aligned}\tag{4.11}$$

where $\breve{\mathrm{a}}$ is an arbitrary orthonormal frame.

The physical meaning of the strain frame $\breve{a}$ is clear from the very definition of the derivative with respect to a vector in conjunction with (4.10) (which is in generally not necessary) for $\boldsymbol{m}=\boldsymbol{a}_i$.

The vectors of the frame $\breve{a}$ are the tangents at the point x to three strained material fibrils issuing from x and mutually orthogonal at the initial time. Moreover, $|\boldsymbol{a}_i|$ is the dilatation coefficient of the material at x at time t in the direction $\boldsymbol{a}_i$.

Restating the last paragraph in Section 4.2, we can say that the specification of the strain frame field permits one to introduce a Riemannian metric on V_t

such that the measurement of angles between deformed material curves and of lengths, areas, and volumes of deformed 1–3-dimensional material objects in this metric gives the values of the respective characteristics for the objects in question prior to strain without resorting to their initial state.

The strain frame is not unique. By taking another orthonormal frame $\check{\mathrm{a}}'$ in the undeformed medium, we obtain another strain frame $\check{a}' := \mathcal{A}\check{\mathrm{a}}'$ by (4.11). Obviously, the frames $\check{a}$ and $\check{a}'$ characterize the same strain.

Note that the values of the expressions (4.6) and (4.10) do not change if the frame $\check{a}$ is replaced by an equivalent frame.

4.4. We take a point x in the deformed medium and a unit vector $\boldsymbol{n}$. The stress per unit geometric area at the point x at time t on the oriented plane with positive normal $\boldsymbol{n}$ will be called the geometric stress (the Cauchy stress [12], or the true stress [13]) and will be denoted by $\boldsymbol{\sigma_n}(x)$. The stress per unit initial area will be referred to as relative (to the initial area).

The geometric measure on V_t coincides with the measure on $V_{t,\check{e}}$ for any orthonormal frame field $\check{e}(x)$ on V_t, and the initial measure, as was already mentioned, coincides with the measure on $V_{t,\check{a}}$.

Generalizing, for an arbitrary nondegenerate frame field $\check{b}(x)$ defined on V_t, we introduce the stress $\boldsymbol{u_n}(x, \check{b})$ per unit area in the Riemannian space $V_{t,\check{b}}$, which will be called the relative stress (relative to $\check{b}(x)$). The symbol $\check{b}$ is omitted if it is clear what frame is meant.

The geometric stress and the stress relative to the initial area are the special cases of $\boldsymbol{u_n}(x, \check{b})$ for $\check{b}(x) = \check{e}(x)$ and $\check{b}(x) = \check{a}(x)$, respectively.

4.5. We introduce the following notation. Let $\boldsymbol{b}_i$ be one of the vectors in the frame $\check{b}$. By $\boldsymbol{n}_i(\check{b})$ we denote the unit normal to the plane spanned by the other two vectors of $\check{b}$; we assume that $\boldsymbol{n}_i(\check{b})$ points to the same side of this plane as $\boldsymbol{b}_i$. Next, $\boldsymbol{u}_i(x, \check{b}) := \boldsymbol{u}_{\boldsymbol{n}_i(\check{b})}(x, \check{b})$ are the relative stresses on the faces of the frame $\check{b}(x)$ at the point (x, t); $\boldsymbol{n}$ is the unit positive normal to a plane Π passing through x; $\mathbf{n}$ is the unit positive normal to the same plane in the metric $V_{t,\check{b}}$ (it will be called the $\check{b}$-normal); $(\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3)^T := \mathbf{n}^{\check{b}}$.

One can prove the generalized Cauchy relation

$$\boldsymbol{u_n}(x, \check{b}) = \boldsymbol{u}_i(x, \check{b})\mathrm{n}_i \tag{4.12}$$

for the relative stresses in completely the same way as the classical Cauchy relation between the Cauchy (geometric) stresses on the plane Π and the faces of $\check{e}$ for an orthonormal frame $\check{e}$.

Introducing the frame

$$\check{u}(x, \check{b}) := \big(\boldsymbol{u}_1(x, \check{b}),\, \boldsymbol{u}_2(x, \check{b}),\, \boldsymbol{u}_3(x, \check{b})\big)^T \tag{4.13}$$

of relative stresses on the faces of $\check{b}(x)$, we can rewrite the dependence (4.12) in the form

$$\boldsymbol{u_n}(x, \check{b}) = \check{u}(x, \check{b})^T \mathbf{n}^{\check{b}}. \tag{4.14}$$

4.6. Let us establish the connection between the relative stress frames for different initial frame fields. We take a point $x \in V_t$ and two arbitrary nondegenerate frames $\check{a}$ and $\check{b}$ at x. (The values of the frames at any other points are

irrelevant.) Suppose that $\check{u}(x,\check{a})$ is known. Let us compute the stress $\boldsymbol{u}_i(x,\check{b})$ on the ith face of $\check{b}$. We denote the unit positive normal, the $\check{a}$-normal, and the $\check{b}$-normal to this face by $\boldsymbol{m}$, $\mathbf{n}$, and $\boldsymbol{n}$, respectively; next, $A := \check{a}^{\check{b}}$, $B := \check{b}^{\check{a}}$, and $\hat{\imath}$ is the column with unit in the ith position and zeros in the remaining positions. Then $\boldsymbol{n} = \boldsymbol{b}_i$ and hence

$$n^{\check{b}} = \hat{\imath}; \tag{4.15}$$

since $\check{a} = A^T\check{b}$, it follows that $\check{b} = A^{-T}\check{a}$ and hence

$$B = A^{-1}. \tag{4.16}$$

By formulas (3.15) and (3.16), the area dilatation coefficient for the transition from the $\check{a}$-measure to the $\check{b}$-measure is equal to

$$K_2(x,\boldsymbol{n}) = \frac{|\check{a}|_{\check{b}}}{|A^T n^{\check{b}}|} \quad \text{and} \quad \mathrm{n}^{\check{a}} = \frac{A^T n^{\check{b}}}{|A^T n^{\check{b}}|}. \tag{4.17}$$

Using (4.17), (4.14), and (4.15), we obtain

$$\boldsymbol{u}_i(x,\check{b}) = K_2^{-1}(x,\boldsymbol{n})\boldsymbol{u}_{\boldsymbol{m}}(x,\check{a}) = \frac{|A^T n^{\check{b}}|}{|\check{a}|_{\check{b}}}\check{u}(x,\check{a})^T \mathrm{n}^{\check{a}} =$$
$$= |\check{a}|_{\check{b}}^{-1}\check{u}(x,\check{a})^T A^T\hat{\imath} = |\check{a}|_{\check{b}}^{-1}\check{u}(x,\check{a})^T(a_{i1},a_{i2},a_{i3})^T = |\check{a}|_{\check{b}}^{-1}\boldsymbol{u}_j(x,\check{a})a_{ij}$$

and since, according to (4.16), the $\check{b}$-volume $|\check{a}|_{\check{b}}$ of the parallelepiped spanned by $\check{a}$ is equal to the determinant $|A| = |B|^{-1}$, we have

$$\check{u}(x,\check{b}) = |B|B^{-1}\check{u}(x,\check{a}). \tag{4.18}$$

This is the desired relation between the relative stresses.

Let us clarify the relationship between the matrice $U_{\check{a}}$ of decomposition of $\check{u}(x,\check{a})$ with respect to the frame $\check{a}$ and matrice $U_{\check{b}}$ of decomposition of $\check{u}(x,\check{b})$ with respect to the frame $\check{b}$. Successively substituting $\check{u}(x,\check{a}) = U_{\check{a}}^T\check{a}$ and $\check{a} = B^{-T}\check{b}$ into (4.18), we have $\check{u}(x,\check{b}) = |B|B^{-1}U_{\check{a}}^T B^{-T}\check{b}$ and hence $U_{\check{b}} = |B|B^{-1}U_{\check{a}}B^{-T}$. Thus the matrix of decomposition of the frame of relative stresses on the faces of the frame $\check{a}$ with respect to $\check{a}$ changes under the replacement of $\check{a}$ by $\check{b}$ as the matrix of some contravariant pseudotensor of rank 2 and weight 1. It will be called the relative stress pseudotensor at the point x.

5 The equilibrium equations with respect to an arbitrary frame field

Suppose that a nondegenerate frame field $\check{b}(x)$ of class C^1 is given in V and an orthonormal frame $\check{e}$ is fixed. Let $\boldsymbol{F}(x)$ and $\boldsymbol{\Psi}(x,\check{b})$ be the mass force density at the point x per unit volume (geometric density) and per unit relative $\check{b}$-volume (relative density), respectively, and let $\boldsymbol{\sigma}_i(x) := \boldsymbol{u}_i(x,\check{e})$ be the Cauchy (geometric) stress at x on the ith face of $\check{e}$.

Since the volume dilatation coefficient (3.9) for the transition from the $\check{b}$-volume to the geometric volume is equal to $|\check{b}|$, we have $\boldsymbol{\Psi}(x,\check{b}) = \boldsymbol{F}(x)|\check{b}|$.

The force and moment equilibrium equations for geometric stresses (e.g., see [13]) read

$$\frac{\partial \boldsymbol{\sigma}_i}{\partial x_i} + \boldsymbol{F}(x) = 0, \quad \Sigma = \Sigma^T, \tag{5.1}$$

where $\Sigma := \check{\sigma}^{\check{e}}$ and the x_i are the coordinates of x in the Cartesian coordinate system with coordinate frame $\check{e}$. By (4.18), we have

$$\check{\sigma} = |\check{b}\,|^{-1} B \check{u}, \tag{5.2}$$

where $B = \check{b}^{\check{e}}$ and hence $\boldsymbol{\sigma}_i(x) = |\check{b}(x)|^{-1} b_{ij}(x) \boldsymbol{u}_j(x,\check{b})$. Therefore,

$$\frac{\partial \boldsymbol{\sigma}_i}{\partial x_i} = \frac{\partial \boldsymbol{\sigma}_i}{\partial \boldsymbol{e}_i} = |\check{b}\,|^{-1} \frac{\partial b_{ij}}{\partial \boldsymbol{e}_i} \boldsymbol{u}_j + b_{ij} \frac{\partial (|\check{b}|^{-1} \boldsymbol{u}_j)}{\partial \boldsymbol{e}_i}. \tag{5.3}$$

The first term in (5.3) is equal to

$$|\check{b}|^{-1} \boldsymbol{u}_j \, div \, \boldsymbol{b}_j. \tag{5.4}$$

Let δ_{ij} be the Kronecker delta, and let b^{ij} be the entries of the matrix B^{-1}. Using the linearity of the derivative with respect to a vector in the vector and the fact that $\check{b} = B^T \check{e}$, i.e.,

$$\check{e} = B^{-T} \check{b} \tag{5.5}$$

and hence $\boldsymbol{e}_i = b^{ki}(x) \boldsymbol{b}_k(x)$, we see that the second term in (5.3) is equal to

$$b_{ij} b^{ki} \frac{\partial (|\check{b}\,|^{-1} \boldsymbol{u}_j)}{\partial \boldsymbol{b}_k} = \delta_{jk} \left(\frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{b}_k} |\check{b}\,|^{-1} - |\check{b}\,|^{-2} \frac{\partial |\check{b}\,|}{\partial \boldsymbol{b}_k} \boldsymbol{u}_j \right) = |\check{b}\,|^{-1} \left(\frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{b}_j} - \frac{\partial \ln |\check{b}\,|}{\partial \boldsymbol{b}_j} \boldsymbol{u}_j \right).$$

By substituting (5.4) into (5.3) and the resulting relation into the first equation in (5.1), we obtain the force equilibrium equation for the relative stresses:

$$\frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{b}_j} + \left(div \, \boldsymbol{b}_j - \frac{\partial \ln |\check{b}|}{\partial \boldsymbol{b}_j} \right) \boldsymbol{u}_j + \boldsymbol{\Psi} = 0. \tag{5.6}$$

Let us compute the matrix $\check{u}^{\check{b}} =: U$. By substituting the expression (5.5) into the equation $\check{\sigma} = \Sigma^T \check{e}$ and then the resulting relation into (5.2), we have $\check{u} = |\check{b}| B^{-1} \Sigma^T B^{-T} \check{b}$. We see that the matrix U is equal to $|\check{b}| B^{-1} \Sigma B^{-T}$ and hence (see (5.1)) is symmetric.

Therefore, the moment equilibrium equation for the relative stresses is also reduced to the symmetry condition for a matrix, in this case, for the matrix U:

$$U = U^T. \tag{5.7}$$

Remark. The arbitrary nondegenerate frame field $\check{b}(x)$ occurring in Eq. (5.6) need not be the coordinate frame field of any curvilinear coordinate system.

6 The elastic strain frame. The elastic state equation

6.1. We momentarily assume that the medium in question is elastic, was in the natural (unstressed) state at the initial time, and was subjected only to elastic strains until time t. In this case, the initial metric of the deformed medium will be referred to as the natural metric; accordingly, the stress $\boldsymbol{u}_{\boldsymbol{n}}(x, \check{a})$ per unit natural area will be called the natural stress (it is also known as the conditional stress vector [13] and the first Piola–Kirchhoff stress vector [12]) and will be denoted by $\boldsymbol{t}_{\boldsymbol{n}}(x)$. The density $\boldsymbol{\Psi}(x, \check{a})$ of mass forces per unit natural volume will also be called the natural density and will be denoted by $\boldsymbol{\Phi}(x)$.

Now the strain frame defined by formula (4.11) completely characterizes the elastic strain of the medium at the point (x, t). It will be called the elastic strain frame.

Since only natural stresses and densities are used in what follows, we usually omit the word "natural."

The Cauchy relation (4.12) passes for the natural stresses into the well-known relation [12, 13]

$$\boldsymbol{t}_{\boldsymbol{n}}(x_0) = \boldsymbol{t}_i(x_0, \check{a})\mathrm{n}_i, \tag{6.1}$$

where $\boldsymbol{t}_i(x_0, \check{a}) := \boldsymbol{t}_{\boldsymbol{n}_i(\check{a})}(x_0)$, $\mathbf{n}$ is the unit $\check{a}$-normal at the point x to the plane orthogonal to $\boldsymbol{n}$, and $(\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3)^T := \mathrm{n}^{\check{a}}$.

Recall that, according to (4.17),

$$\mathrm{n}^{\check{a}} = \frac{A^T n^{\check{e}}}{|A^T n^{\check{e}}|}, \quad A = \check{a}^{\check{e}} \tag{6.2}$$

for $\check{b} = \check{e}$.

We assume that the medium in the natural state is homogeneous and isotropic and that the stress at an arbitrary point x of the deformed medium is completely determined by strain at x, i.e., by the elastic strain frame $\check{a}(x)$.

Using the frame $\check{t}(x, \check{a})$ of natural stresses at the point x on the faces of the frame $\check{a}(x)$, we introduce the stress matrix $\hat{t}(x, \check{a}) := (t_{ij}(x, \check{a})) := \check{t}^{\check{a}}(x, \check{a})$. Under our assumptions about the medium, the strain frame $\check{a}(x)$ determines the stress matrix by the following chain of relations [12]: $\check{a} \to \hat{\varepsilon}^e(\check{a}) := (\varepsilon^e_{ij}(\check{a})) := (\frac{1}{2}(\langle \boldsymbol{a}_i, \boldsymbol{a}_j\rangle - \delta_{ij}))$ (the Cauchy–Lagrange elastic strain matrix); $\hat{\varepsilon}^e \to \hat{f}(\hat{\varepsilon}^e) := (f_{ij}(\hat{\varepsilon}^e))$, where f_{ij} are functions of class C^1 characterizing the elastic properties of the medium; $\hat{t}(x, \check{a}(x)) = \hat{f}(\hat{\varepsilon}^e(\check{a}(x)))$.

Thus the equations of the elastic state of the medium can be represented in the form

$$\hat{t}(x, \check{a}(x)) = \hat{f}(\hat{\varepsilon}^e(\check{a}(x))), \tag{6.3}$$

or

$$\boldsymbol{t}_i(x, \check{a}(x)) = f_{ji}(\hat{\varepsilon}^e(\check{a}(x)))\boldsymbol{a}_j(x), \quad i = 1, 2, 3. \tag{6.4}$$

Thus the matrix $\hat{t}(x, \check{a})$, which is a special case of U (see Section 5), is symmetric, and the principal stress frame is coaxial with the principal strain frame.

Since in our case the functions $\hat{t}$, $\check{t}$, and $\boldsymbol{t}_i$ depend only on the frame $\check{a}(x)$, the first argument x of these functions can be omitted in what follows.

6.2. Now consider an elastoplastic medium in the process of elastoplastic deformation. First, we assume that, at any time t of the process, the medium, or at least a sufficiently small neighborhood of any point of the medium, can be unloaded to a natural state by an elastic deformation. We treat this natural state and the state at time t as the initial undeformed state and the state obtained from it by an elastic deformation, respectively. Then (see 6.1) there exists an elastic strain frame at time t at each point of the medium.

There also exists a similar frame field under more general assumptions. In general, even a small neighborhood of a point cannot be unloaded to an unstressed state. There always remain residual stresses. However, one can assume that these stresses tend to zero as the neighborhood shrinks into the point, and so the unloaded states tend to an unstressed state.

We take an arbitrary point x of the deformed medium at time t. Let v be a neighborhood of x. Let us unload the part of the medium contained in v and consider the strain frame $\check{a}_v(x,t)$ at x, corresponding to the displacement of the unloaded part to its position at time t. Suppose that there exists a limit $\lim \check{a}_v(x,t) =: \check{a}(x,t)$ as $v \to x$ and that the field $\check{a}(x,t)$ is continuous.

Now consider an arbitrary material curve in the deformed medium. We cover it by finitely many balls of radius r and cut into finitely many partial curves each of which entirely lies in one of the balls. By unloading the part of the medium lying in a ball, we obtain the "unloaded" length of the corresponding partial curve. We define the "unloaded" length of the entire curve as the sum of unloaded lengths of its parts.

One can show that the unloaded lengths of the curve tend as $r \to 0$ to a limit, which will be called the "natural" length of the material curve in question. The natural measures of material surfaces, volumes, and angles can be defined in a similar way. It turns out that the natural measure defined thus constructed for material objects coincides with their measure in the Riemannian space generated on V_t by the frame field $\check{a}(x,t)$.

Thus in both cases at any time t at each point of the medium there exists a frame determining how the measure of 1–3-dimensional material objects at time t has changed compared with their natural measure. It is this property of the frame that is important in what follows.

Therefore, let us introduce a notion of elastic strain frame for elastoplastic media generalizing the corresponding notion for elastic media and based neither on any concept of actual motion of the medium nor on any assumptions related to full or partial unloading of parts of the medium.

Suppose that at each time the medium has some "elastic structure", which permits one to judge about the natural measure near each point of the medium. More precisely, we shall adopt the following assumption.

Assumption. For each given time t, to each point x of the medium one can assign a nondegenerate frame $\check{a}(x,t)$ (which will be called the elastic strain frame of the medium at the point (x,t)) such that the Riemannian metric generated in V_t by the field $\check{a}(x,t)$ (the metric of the space $V_{t,\check{a}}$) is "natural."

Remark.

1. The "naturality" of a measure, which is clear if full unloading is possible, is defined as certain relationships, postulated below, with the stress and the plastic strain.

2. In a polycrystal material, crystal lattices can serve as a prototype of an "elastic structure."

The elastic strain frame field on V_t is not unique. Indeed, if $\breve{a}(x,t)$ is an elastic strain frame field, then for a frame field $\breve{a}'(x,t)$ to be an elastic strain frame field it is necessary and sufficiently that these fields be equivalent, i.e., that the frames $\breve{a}(x,t)$ and $\breve{a}'(x,t)$ be equivalent for all $x \in V_t$.

Indeed, if the fields $\breve{a}$ and $\breve{a}'$ are equivalent, then these frames generate equal scalar products at each point. Consequently, the Riemannian metrics generated by the fields $\breve{a}$ and $\breve{a}'$ coincide; i.e., $\breve{a}'$ also generates a natural metric and hence is an elastic strain frame field. The necessity can also be easily justified.

The following lemma will be often used in the sequel.

Lemma. For nondegenerate frames $\breve{a}$ and $\breve{a}'$ to be equivalent, it is necessary and sufficiently that there be an orthogonal 3×3 matrix Q such that

$$\breve{a}' = Q\breve{a}. \tag{6.5}$$

Indeed, let the frames be equivalent: $\forall \boldsymbol{\alpha},\, \boldsymbol{\beta} \quad \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\breve{a}'} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\breve{a}}$.

We denote $Q := (\breve{a}'^{\breve{a}})^T$; then $\breve{a}' = Q\breve{a}$. Since each of the frames is orthogonal with respect to the scalar product generated by it, it follows that $\delta_{ij} = \langle \boldsymbol{a}'_i, \boldsymbol{a}'_j \rangle_{\breve{a}'} = \langle \boldsymbol{a}'_i, \boldsymbol{a}'_j \rangle_{\breve{a}} = \langle q_{ik}\boldsymbol{a}_k, q_{jl}\boldsymbol{a}_l \rangle_{\breve{a}} = q_{ik}q_{jl}\langle \boldsymbol{a}_k, \boldsymbol{a}_l \rangle_{\breve{a}} = q_{ik}q_{jl}\delta_{kl}$ for any i and j, i.e., $q_{ik}q_{jk} = \delta_{ij}$ and hence Q is orthogonal. The sufficiency can be proved even easier.

7 The plastic strain tensor

To characterize plastic strain, we need additional information on the change in the measure of geometric objects undergoing deformation.

7.1. The measure dilatation coefficients for the mapping of $\mathrm{V}_{\breve{\alpha}}$ *into* $V_{\breve{\beta}}$. Here the situation is the same as in Sections 4.2–4.3; $\mathrm{V}_{\breve{\alpha}}$ plays the role of the domain of the initial position of the medium, $V_{\breve{\beta}}$ plays the role of the domain V_t of the final position of the medium, $\breve{\alpha}(x)$ is used instead of $\breve{\mathrm{a}}(\mathrm{x})$, and the $\breve{\beta}$-measure is used instead of the geometric measure in V_t. We introduce the frame

$$\breve{a}(x) := \frac{\partial x(\mathrm{x})}{\partial \breve{\alpha}(\mathrm{x})},$$

which plays the role of $\breve{a}(x)$ in 4.2–4.3. Arguing as in 4.2–4.3, we arrive at the conclusion that the $\breve{a}$ - measure of objects in V coincides with the $\breve{\alpha}$-measure of their preimages in V. But then the following assertion holds.

Assertion. The measure dilatation coefficients comparing the $\breve{\alpha}$-measure of 1–3-dimen-sional objects in V *with the $\breve{\beta}$-measure of their images in V are given by formulas (3.7), (3.9), and (3.16) with $\breve{\alpha}$ replaced by $\breve{a}$.*

For the $\breve{\alpha}$-measure of the initial angles, using the reasoning and notation of Section 4.2, we have

$$\cos(\widehat{\mathbf{m}_1, \mathbf{m}_2})_{\mathrm{x},\breve{\alpha}} = \frac{\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle_{\breve{a}(x)}}{|\boldsymbol{m}_1|_{\breve{a}(x)} \cdot |\boldsymbol{m}_2|_{\breve{a}(x)}}, \tag{7.1}$$

where $(\widehat{\mathbf{m_1}, \mathbf{m}_2})_{\mathrm{x},\breve{\alpha}}$ is the angle between $\mathbf{m_1}$ and $\mathbf{m_2}$ at the point x in the $\breve{\alpha}$-metric.

7.2. In what follows, we consider a medium that moves during the time interval $[t_b, t_e] =: T$ and is elastoplastic (in the sense of Section 6.2) at each time $t \in T$. All earlier-introduced assumptions and the main notation are preserved; we only introduce an additional variable, time (t), which occurs as an argument of the functions specifying the law of motion of the medium, the elastic strain frame, the velocities $\boldsymbol{v}(x,t)$ of points of the medium, etc.

Let $W := \{(x,t) | \, t \in T, \, x \in V_t\}$. Let $\breve{a}(x,t)$ be an elastic strain frame field of the medium of the class $C^1(W)$. We take an arbitrary $t_0 \in T$ and denote the law of motion of the medium starting from time t_0 by

$$x = x_{t_0}(\mathrm{x}, t). \tag{7.2}$$

To each point x in space, this law assigns the spatial position x at time $t \in T$ of the material point residing at x at time t_0. We shall assume that $x_{t_0} \in C^2(V_{t_0} \times T)$.

The comparison of natural measures of material objects at time t and t_0 characterizes the plastic strain of the medium on the interval $[t_0, t]$.

Just as the frame field $\partial x/\partial \breve{\mathrm{a}}$ on V_t, where $\breve{\mathrm{a}}$ defines the geometric measure at the initial time (since $E = E_{\breve{\mathrm{a}}}$), completely characterizes the change in the geometric measure of material objects (see Section 4.3), so the frame field

$$\breve{b}(x,t) := \frac{\partial x_{t_0}(\mathrm{x},t)}{\partial \breve{a}(\mathrm{x},t_0)}, \quad (x = x_{t_0}(\mathrm{x},t)), \tag{7.3}$$

where $\breve{a}(x, t_0)$ defines the natural metric on V_{t_0} at time t_0, completely characterizes the change in the natural measure of material objects in time $[t_0, t]$, i.e., the plastic strain on this time interval (the elastic strain frame field $\breve{a}(x,t)$ on V_t and V_{t_0} being known).

Indeed, we take $\forall \mathrm{x}_0 \in V_{t_0}$ and $x_0 := x_{t_0}(\mathrm{x}_0, t)$ and momentarily set $\breve{a}(\mathrm{x}_0, t_0) =: \breve{a}_0 =: (\boldsymbol{a}_1^0, \boldsymbol{a}_2^0, \boldsymbol{a}_3^0)^T$, $\breve{a}(x_0, t) =: \breve{a}$, and $\breve{b}(x_0,t) =: \breve{b}$. According to the Assertion in Section 7.1 (with $\breve{a}$ as $\breve{\beta}$ and $\breve{b}$ as $\breve{a}$), the natural length dilatation coefficient at a point (x_0, t) in an arbitrary direction $\boldsymbol{m}$ is equal to

$$\tilde{K}_1(x_0, t) = \frac{|\boldsymbol{m}|_{\breve{a}}}{|\boldsymbol{m}|_{\breve{b}}}. \tag{7.4}$$

Moreover, for two arbitrary nonzero vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ at x_0, one can take oriented material curves for which these vectors are the positive tangent vectors at the point x_0 at time t and introduce the positive tangent vectors $\mathbf{m}_i$ to these curves at time t_0 at the point x_0.

Now, using the properties of the elastic strain frame $\check{a}$ (see the assumption in Section 6, (4.6), and (7.1) with $\check{\alpha} = \check{a}_0$), for the material angle $(\widehat{\boldsymbol{m}_1, \boldsymbol{m}_2})_{x_0,t}$ and the corresponding $(\widehat{\mathbf{m}_1, \mathbf{m}_2})_{\mathrm{x}_0,t_0}$ we have

$$\begin{aligned} \cos(\boldsymbol{m}_1\hat{,}\boldsymbol{m}_2)_{x_0,t} &= \frac{\langle \boldsymbol{m}_1, \boldsymbol{m}_2\rangle_{\check{a}}}{|\boldsymbol{m}_1|_{\check{a}}|\boldsymbol{m}_2|_{\check{a}}} \\ \cos(\mathbf{m}_1\hat{,}\mathbf{m}_2)_{\mathrm{x}_0,t_0} &= \frac{\langle \mathbf{m}_1, \mathbf{m}_2\rangle_{\check{a}_0}}{|\mathbf{m}_1|_{\check{a}_0}|\mathbf{m}_2|_{\check{a}_0}} = \frac{\langle \boldsymbol{m}_1, \boldsymbol{m}_2\rangle_{\check{b}}}{|\boldsymbol{m}_1|_{\check{b}}|\boldsymbol{m}_2|_{\check{b}}} \end{aligned} \tag{7.5}$$

Thus the specification of $\check{b}$ for known $\check{a}$ permits one to compute the natural length dilatation coefficient at x_0 at time t in any direction $\boldsymbol{m}$ (see (7.4)) and, for any material curves issuing from x_0 at time t, compute the cosines of the natural angles between them at the point x_0 at time t and at the point x_0 at time t_0 (see (7.5)).

In particular, for the vectors $\boldsymbol{b}_i$ of the frame $\check{b}$, we have $b_i^{\check{b}} = \hat{i}$ and $|\boldsymbol{b}_i|_{\check{b}} = 1$. Hence (see (7.4) and (7.5)) $|\boldsymbol{b}_i|_{\check{a}}$ is the material length dilatation coefficient along $\boldsymbol{b}_i$ at the point x_0 at time t; moreover, $\cos(\widehat{\boldsymbol{b}_i, \boldsymbol{b}_j})_{x_0,t} = \langle \boldsymbol{b}_i, \boldsymbol{b}_j\rangle_{\check{a}}/(|\boldsymbol{b}_i|_{\check{a}}|\boldsymbol{b}_j|_{\check{a}})$, while the cosine of the material angle between the corresponding vectors $\boldsymbol{a}_i^0$ and $\boldsymbol{a}_j^0$ of the frame $\check{a}_0$ at x_0 at time t_0 (see (7.5)) is zero for $i \neq j$.

We see that the frame $\check{b}$ plays the same role in determining the plastic strain of the medium in time $[t_0, t]$ as $\check{a}$ plays in determining the elastic strain.

By analogy with the Cauchy–Lagrange elastic strain matrix, we introduce the plastic strain matrix at a point x_0 from time t_0:

$$\hat{\varepsilon}^p := \left(\tfrac{1}{2}(\langle \boldsymbol{b}_i, \boldsymbol{b}_j\rangle_{\check{a}} - \delta_{ij})\right) =: \frac{\hat{\gamma} - 1}{2}. \tag{7.6}$$

By introducing the matrix $B := \check{b}^{\check{a}}$ and by taking into account the definition of the scalar product $\langle\cdot,\cdot\rangle_{\check{a}}$, we obtain

$$\hat{\gamma} = B^T B.$$

The physical meaning of $\hat{\varepsilon}^p$ is the same as that of $\hat{\varepsilon}^e$ with the only difference that geometric lengths and angles are replaced by natural ones. Let us study how $\hat{\varepsilon}^p$ changes if we replace the elastic strain frame $\check{a}_0$ at the point (x_0, t_0) by the equivalent frame $\check{a}_0'$. It suffices keep track of the change in $\hat{\gamma}$. We denote $Q_0 := ((\check{a}_0')^{\check{a}_0})^T$; then

$$\check{a}_0' = Q_0 \check{a}_0. \tag{7.7}$$

For the objects $\check{b}'$, B', $(\hat{\varepsilon}^p)'$, and $\hat{\gamma}'$ corresponding to $\check{a}_0'$ to belong to the class C^1, we replace the field $\check{a}(x,t)$ by an equivalent field $\check{a}'(x,t)$ such that

$$\check{a}'(x,t) = Q(x,t)\check{a}(x,t), \tag{7.8}$$

where $Q(x,t)$ is a C^1 field of orthogonal matrixes (see the lemma in Section 6) and $Q(\mathrm{x}_0, t_0) = Q_0$. Now, in view of (7.3), (7.7) and (7.8), we obtain

$$\check{b}' = \frac{\partial x_{t_0}(\mathrm{x}_0, t)}{\partial \check{a}'(\mathrm{x}_0, t_0)} = \frac{\partial x_{t_0}(\mathrm{x}_0, t)}{\partial \mathrm{x}} \check{a}_0' = Q_0 \check{b} = Q_0 B^T \check{a} = Q_0 B^T Q^T \check{a}'.$$

It follows that

$$B' = QBQ_0^T. \tag{7.9}$$

Therefore, $\hat{\gamma}' = B'^T B' = Q_0 B^T B Q_0^T = Q_0 \hat{\gamma} Q_0^T$; i.e., $\hat{\gamma}$ is the matrix of some covariant rank two tensor $\boldsymbol{\gamma}$ in the frame $\check{a}_0$. This tensor is defined at (x_0, t_0) for elastic strain frames and depends on time.

Hence the same it true for $\hat{\varepsilon}^p$ and for the tensor $\boldsymbol{\varepsilon}^p$ of plastic strain at the point x_0 from time t_0.

In general, the present paper deals only with rank two tensors, and since all of them are associated only with elastic strain frames, it follows that the matrices of transformation from one frame to another are orthogonal, so that the tensors are covariant.

Remark. The tensor $\boldsymbol{\varepsilon}^p$ is different from the tensor, traditionally denoted by the same letter, defined as the difference between the full and elastic strain tensors or reckoned from the initial time.

Let us introduce the plastic strain rate tensor $\boldsymbol{\zeta}(\mathrm{x}_0, t_0)$ at the point (x_0, t_0). By definition, it is equal to the total t-derivative at time t_0 of the plastic strain tensor at the point x_0 reckoned from time t_0,

$$\boldsymbol{\zeta}(\mathrm{x}_0, t_0) = (\mathrm{d}\boldsymbol{\varepsilon}^p/\mathrm{d}t)_{|t_0}. \tag{7.10}$$

8 The strain consistency equation

On the mass point trajectory passing through x_0 at time t_0, we have

$$\frac{\partial x_{t_0}(\mathrm{x}_0, t)}{\partial \mathrm{x}} \check{a}(\mathrm{x}_0, t_0) = (B(x,t))^T \check{a}(x,t),\ x = x_{t_0}(\mathrm{x}_0, t)$$

according to (7.3). By taking the total t-derivative of this identity at $t = t_0$ and by changing the order of differentiation on the left-hand side, we obtain

$$\frac{\partial \boldsymbol{v}(\mathrm{x}_0, t_0)}{\partial \check{a}(\mathrm{x}_0, t_0)} = \left[\frac{\mathrm{d}}{\mathrm{d}t} B^T(x,t)\right]_{|t_0} \check{a}(\mathrm{x}_0, t_0) + B^T(\mathrm{x}_0, t_0)\left[\frac{\mathrm{d}}{\mathrm{d}t}\check{a}(x,t)\right]_{|t_0}.$$

Taking into account the fact that $B^T(\mathrm{x}_0, t_0) = I$ and passing to contracted notation, we obtain the equation

$$\frac{\partial \boldsymbol{v}}{\partial \check{a}} = \left(\frac{\mathrm{d}B^T}{\mathrm{d}t}\right)_{|t_0} \check{a} + \left(\frac{\mathrm{d}\check{a}}{\mathrm{d}t}\right)_{|t_0}, \tag{8.1}$$

satisfied at the point (x_0, t_0) by any elastic strain frame field $\check{a}$ and by the corresponding B (since we deal with a specific motion of the medium, the field $\boldsymbol{v}$ is fixed).

By taking the total t-derivative of (7.9), for $t = t_0$ we obtain the law describing how the matrix $\mathrm{d}B/\mathrm{d}t_{|t_0}$ changes under the passage to an arbitrary elastic strain frame field $\check{a}'(\mathrm{x}, t)$ (see (7.8)):

$$\frac{\mathrm{d}B'}{\mathrm{d}t}_{|t_0} = \frac{\mathrm{d}Q}{\mathrm{d}t}_{|t_0} Q_0^T + Q_0 \frac{\mathrm{d}B}{\mathrm{d}t}_{|t_0} Q_0^T. \tag{8.2}$$

For this law to be a tensor law on W, it is necessary and sufficient that

$$\frac{\mathrm{d}Q(\mathrm{x}_0, t_0)}{\mathrm{d}t} = 0 \ \ \forall (\mathrm{x}_0, t_0) \in W, \tag{8.3}$$

i.e. that $Q(x,t)$ be constant along the space-time trajectories of the points of the medium.

We say that elastic strain frame fields $\breve{a}$ and $\breve{a}'$ related by formula (7.8) with Q satisfying (8.3) are kinematically equivalent.

Thus the set of elastic strain frame fields splits into classes of kinematically equivalent fields, and the matrix $dB/dt_{|t_0}$ varies according to the tensor law on each of the classes.

These classes are sufficiently ample; namely, the following assertion is valid: if $\breve{a}(x,t)$ is an elastic strain frame field of the medium in W, then for each t_0 and each elastic strain frame field $\breve{a}_0(x)$ given at time t_0 on V_{t_0} there exists a field $\breve{a}^*(x,t)$ kinematically equivalent to $\breve{a}(x,t)$ and coinciding with $\breve{a}_0(x)$ at time t_0.

Indeed, let $Q_0(x)$ be the field of orthogonal matrices such that $\breve{a}_0(x) = Q_0(x)\breve{a}(x,t_0)$. We introduce a field of orthogonal matrices on W by setting $Q(x,t_0) = Q_0(x)$ and by requiring that $Q(x,t)$ does not vary along the trajectories of the points of the medium. Let $\breve{a}^*(x,t) := Q(x,t)\breve{a}(x,t)$; then first, $\breve{a}^*(x,t_0) := Q(x,t_0)\breve{a}(x,t_0) = \breve{a}_0(x)$ and second, the field $\breve{a}^*$ is kinematically equivalent to $\breve{a}(x,t)$, since $dQ(x,t)/dt = 0$.

Assumption. For the medium, there exists an elastic strain frame field of the class $C^1(W)$ for which the matrix $dB/dt_{|t_0}$ is symmetric on W. We say that this field is kinematically constrained.

Remark 8.1. It is shown in the paper mentioned second in the footnote in Section 1 that if the law of motion of the medium is of the class C^3, then the existence of a kinematically constrained field automatically follows from the existence of at least one elastic strain frame field of the class C^2.

It turns out that the set of kinematically constrained elastic strain frame fields is a class of kinematically equivalent elastic strain frame fields.

Indeed, if $\breve{a}$ is a kinematically constrained elastic strain frame field and $\breve{a}'$ is a kinematically equivalent field, then $(dB'/dt)_{|t_0} = Q_0(dB/dt)_{|t_0}Q_0^T$ by (8.2), and hence this matrix is symmetric (since so is $dB/dt_{|t_0}$); thus $\breve{a}'$ is also kinematically constrained. On the other hand, if elastic strain frame fields $\breve{a}$ and $\breve{a}'$ are kinematically constrained, then the matrices $dB/dt_{|t_0}$ and $dB'/dt_{|t_0}$ are symmetric, and hence so is $\left(dQ/dt_{|t_0}\right)Q_0^T$ (see (8.2)). Consequently, $(dQ/dt)_{|t_0}Q_0^T = Q_0(dQ^T/dt)_{|t_0}$, and hence

$$\frac{\mathrm{d}Q}{\mathrm{d}t}_{|t_0} Q_0^T = \frac{1}{2}\left(\frac{\mathrm{d}Q}{\mathrm{d}t}_{|t_0} Q_0^T + Q_0\frac{\mathrm{d}Q^T}{\mathrm{d}t}_{|t_0}\right) = \frac{1}{2}\frac{\mathrm{d}(QQ^T)}{\mathrm{d}t}_{|t_0} = 0,$$

since $QQ^T = I$. Thus $dQ/dt_{|t_0} = 0$; i.e., $\breve{a}$ and $\breve{a}'$ are kinematically equivalent.

The class of kinematically constrained elastic strain frame fields plays a special role. Let $\breve{a}$ be a kinematically constrained elastic strain frame field, and let (x_0, t_0) be an arbitrary point in W. Taking into account the symmetry of

$dB/dt_{|t_0}$, the relation $B_{|t_0} = I$, and formulas (7.10) and (7.6), we obtain the following expression for the matrix $\boldsymbol{\zeta}(\mathrm{x}_0, t_0)$ in the frame $\breve{a}_0$:

$$\zeta_{\breve{a}_0}(\mathrm{x}_0, t_0) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varepsilon}^p\right)_{|t_0}\right]_{\breve{a}_0} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varepsilon}^p\right)_{|t_0} = \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(B^T B - I)\right]_{|t_0} =$$
$$= \frac{1}{2}\left[\left(\frac{\mathrm{d}B^T}{\mathrm{d}t}B\right)_{|t_0} + \left(B^T\frac{\mathrm{d}B}{\mathrm{d}t}\right)_{|t_0}\right] = \frac{\mathrm{d}B^T}{\mathrm{d}t}_{|t_0}$$

Substituting this into (8.1), we obtain the strain rate consistency equation

$$\frac{\partial \boldsymbol{v}}{\partial \breve{a}} = \zeta_{\breve{a}}\breve{a} + \frac{\mathrm{d}\breve{a}}{\mathrm{d}t}, \tag{8.4}$$

which is satisfied in W by each kinematically constrained elastic strain frame field.

The physical meaning of Eq. (8.4) becomes clear if we remember how (8.1) was obtained: the material rate $\partial\boldsymbol{v}(x,t)/\partial\breve{a}(x,t)$ due to the motion of the medium of the frame coinciding with $\breve{a}(x,t)$ at the point (x,t) at time t is equal to the material rate $d\breve{a}(x,t)/dt$ of the elastic strain frame plus a correction term depending on the matrix of the plastic strain rate tensor in the frame $\breve{a}$.

Remark 8.2. For "small" elastic strains, more precisely, under the assumption that there exists an orthonormal frame $\breve{e}$ and a number $\delta << 1$ such that $|\boldsymbol{a}_i - \boldsymbol{e}_i|$ and $|\partial\boldsymbol{a}_i/\partial x_j|$, $|\partial\boldsymbol{a}_i/\partial t|$, $|\partial\boldsymbol{v}/\partial x_i| \le \delta$, it follows from (8.1), neglecting quantities of the order of δ^2, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varepsilon}_{t_0}(x_0, t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varepsilon}^p_{t_0}(x_0, t_0) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varepsilon}^e(x_0, t_0), \quad \forall (x_0, t_0) \in W,$$

where the full and plastic strains are reckoned from time t_0. It is this equation that essentially substitutes the traditional equation $\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^p + \mathrm{d}\boldsymbol{\varepsilon}^e$. (Although this substitution is not justified, it leads to correct results in our situation.)

9 The elastic and plastic constitutive equations for elastoplastic media "homogeneous and isotropic in the natural state"

We understand the media specified in the title as media possessing the following properties.

Assumption 9.1. Under elastoplastic deformation, the elastic strain frames and the natural stresses are related by the same elastic constitutive equation (6.3)≡(6.4) as in the case of elastic media homogeneous and isotropic in the natural state.

Let $\breve{a}^*$ be the principal elastic strain frame at a point $(x,t) \in W$; then, by Assumption 9.1, the principal frame $\check{t}(\breve{a}^*) = (\boldsymbol{t}_1(\breve{a}^*), \boldsymbol{t}_2(\breve{a}^*), \boldsymbol{t}_3(\breve{a}^*))^T$ of natural stresses on the faces of $\breve{a}^*$ and the frame $\breve{a}^*$ are coaxial; here $\boldsymbol{t}_i(\breve{a}^*)$ are the principal natural stresses. The only possible nonzero coordinate $\boldsymbol{t}_i(\breve{a}^*)$ in the frame $\breve{a}^*$ is $t_{\underline{ii}}(\breve{a}^*) =: t_i(\breve{a}^*)$ will also be referred to as the principal stress, which, in view of our notation, will not lead to a misunderstanding; $(t_1(\breve{a}^*), t_2(\breve{a}^*), t_3(\breve{a}^*)) =: \mathrm{t}(\breve{a}^*)$.

Assumption 9.2. If $\breve{a}^*$ is the principal elastic strain frame of the medium at a point (x,t), then the tensor $\boldsymbol{\zeta}(x,t)$ is coaxial to $\breve{a}^*$ for all $(x,t) \in W$; moreover, there exist known functions $p_i : \mathbf{R}^3 \to \mathbf{R}$ $(i = 1,2,3)$, describing the plastic properties of the medium such that

$$\zeta_{\breve{a}^*}(x,t) = \left\| \begin{array}{ccc} p_1(\mathrm{t}(\breve{a}^*)) & 0 & 0 \\ 0 & p_2(\mathrm{t}(\breve{a}^*)) & 0 \\ 0 & 0 & p_3(\mathrm{t}(\breve{a}^*)) \end{array} \right\| =: P(\mathrm{t}(\breve{a}^*)). \tag{9.1}$$

Remark. 1. Thus the medium in question is so far "perfectly plastic" in the sense that the plastic strain rates are independent of the strain history and depend only on the principal stresses.

2. The assumption that $\zeta_{\breve{a}^*}$ depends only on the principal stresses and is independent of the stress increments seems to us to be quite natural for "slow" deformations.

3. Using the technique applied in the study of the yield locus of an isotropic material (e.g., see [14]), one can readily show that

$$p_3(t_2, t_3, t_1) = p_2(t_3, t_1, t_2) = p_1(t_1, t_2, t_3) =: p(\mathrm{t}) \quad \forall \mathrm{t}.$$

For an arbitrary elastic strain frame $\breve{a}$ at a point (x,t), by R we denote the orthogonal matrix such that $\breve{a} = R\breve{a}^*$. Then $\zeta_{\breve{a}} = R\zeta_{\breve{a}^*}R^T$, and the plastic constitutive equation of the medium (9.1) can be represented in the form

$$\zeta_{\breve{a}}(x,t) = RP(\mathrm{t}(\breve{a}^*))R^T \quad \forall (x,t) \in W. \tag{9.2}$$

10 The system of elastoplastic strain equations in the Eulerian variables

Let us summarize our study. We consider an elastoplastic medium satisfying the assumptions in Sections 6 and 8 and moving in a space-time domain W. The natural density $\boldsymbol{\Phi}(x,t)$ of applied mass forces of the class $C(W)$ and the density ρ_0 of the undeformed medium are known. The unknowns are the velocity field $\boldsymbol{v}(x,t)$ of the points of the medium and any of the kinematically equivalent kinematically constrained elastic strain frame fields $\breve{a}(x,t)$ of the medium, where $\breve{a}$ and $\boldsymbol{v}$ are related by two equations, namely, the strain consistency equation (8.4) (or (8.1) is $\breve{a}$ not kinematically constrained) and the equation of motion obtained from the force equilibrium equation (5.6) by the substitution of $\breve{a}(x,t)$ for $\check{b}(x)$. Here the relative stresses $\boldsymbol{u}_i(x)$ become the natural stresses

$\boldsymbol{t}_i(x,t)$, and the mass forces include also inertial forces; i.e., $\boldsymbol{\Psi(x)}$ is replaced by $\boldsymbol{\Phi}(x,t) - \rho_0(d\boldsymbol{v}/dt)$.

These equations are valid for any media with an "elastic structure." Along with the unknowns $\breve{a}$ and $\boldsymbol{v}$, they also contain the natural stress frame $\check{t}$ and a matrix describing the plastic strain; therefore, to obtain a closed system, one should supplement these equations with equations describing the elastic and plastic properties of the material.

For media homogeneous and isotropic in the undeformed state, i.e., satisfying Assumptions 9.1 and 9.2, the supplementing equations are the elastic and plastic constitutive equations (6.4) and (9.2).

Thus the system of dynamic equations of elastoplastic strain in this case has the form

$$\begin{aligned} &\rho_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial \boldsymbol{t}_i}{\partial \boldsymbol{a}_i} + \left(\mathrm{div}\boldsymbol{a}_i - \frac{\partial \ln |\breve{a}|}{\partial \boldsymbol{a}_i}\right)\boldsymbol{t}_i + \boldsymbol{\Phi} \\ &\frac{\mathrm{d}\breve{a}}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}}{\partial \breve{a}} - RPR^T\breve{a}, \ \ \forall (x,t) \in W \end{aligned} \tag{10.1}$$

$$\boldsymbol{t}_i = f_{ji}(\hat{\varepsilon}^e(\breve{a}))\boldsymbol{a}_j. \tag{10.2}$$

The total derivatives in (10.1) are expressed via $\boldsymbol{v}$ and $\breve{a}$ as follows:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{v}} + \frac{\partial \boldsymbol{v}}{\partial t}, \ \frac{\mathrm{d}\breve{a}}{\mathrm{d}t} = \frac{\partial \breve{a}}{\partial \boldsymbol{v}} + \frac{\partial \breve{a}}{\partial t}. \tag{10.3}$$

System (10.1) (in view of (10.2)) is a system of a vector equation and a frame equation for the vector function $\boldsymbol{v}$ and the frame function $\breve{a}$. In vector form, it consists of four equations for $\boldsymbol{v}$ and $\boldsymbol{a}_i$ $(i = 1,2,3)$.

The equations are represented in coordinate-free form. After the introduction of some coordinate system and the replacement of derivatives along vectors by their expressions via partial derivative with respect to coordinates, one obtains a system of quasilinear first-order partial differential equations.

Equations (10.1) are solved for the time derivatives of the desired functions, which makes the system suitable for numerical time-marching solution.

If the functions p_i describing the plastic properties of the medium depend not only on the principal stresses but also on parameters determined by the strain history, then one should supplement the system with equations relating these parameters to the desired functions.

In the absence of plastic strains ($p_i = 0$), system (10.1) passes into system (41) of elasticity equations.[2]

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